Geometrical Interpretation of the Derivative

(2.3) Let f be a differentiable function given by the equation $y = f(x)$.

On the graph of $y = f(x)$, let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ be distinct points near to each other.

If β is the angle that the secant line PQ makes with the x-axis, then

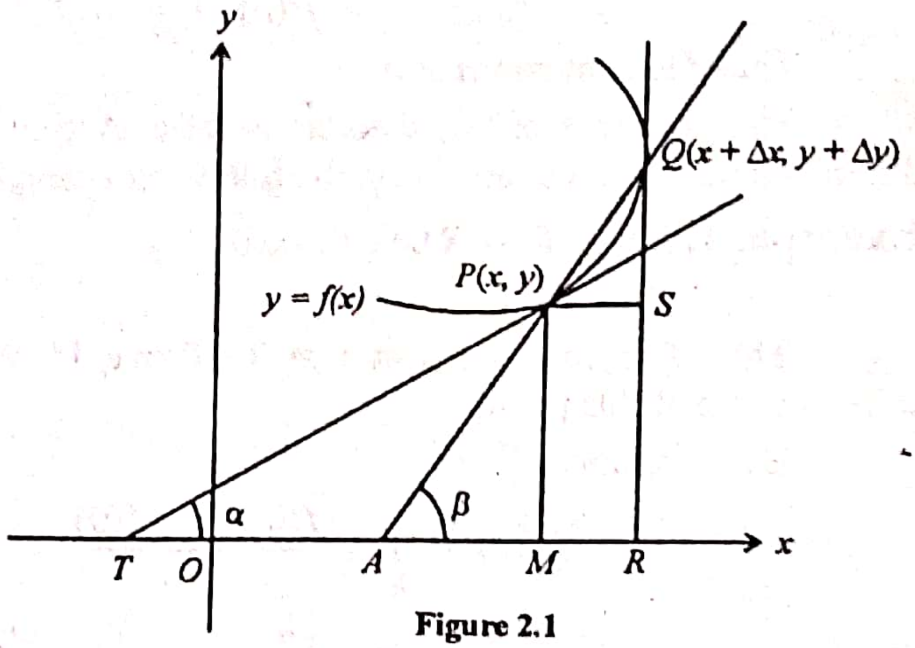


Figure 2.1

$$\tan\beta = \frac{QS}{PS} = \frac{(y + \Delta y) - y}{(x + \Delta x) - x}$$

$$= \frac{f(x + \Delta x) - f(x)}{x + \Delta x - x}$$

$$= \frac{\Delta y}{\Delta x} = \text{Slope of the secant line } APQ. \quad (1)$$

As Δx approaches 0, the point Q moving along the graph of $y = f(x)$ approaches P, the chord PQ approaches the tangent line PT in its limiting position and measure β of angle MAQ approaches $\alpha = m\angle OTP$. Hence taking limits as $\Delta x \to 0$, (1) becomes

$$\tan\alpha = \frac{dy}{dx} = m$$

i.e., the derivative of the function f at the point P represents the slope of the tangent line to the curve $y = f(x)$ at that point.

2.2 Differentiation Rules

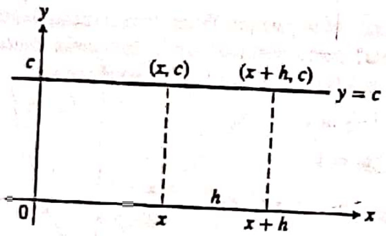
This section shows how to differentiate functions without having to apply the definition each time.

Powers, Multiples, Sums, and Differences

The first rule of differentiation is that the derivative of every constant function is zero.

> **Rule 1 Derivative of a Constant**
>
> If c is constant, then $\frac{d}{dx}c = 0$.

EXAMPLE 1 $\frac{d}{dx}(8) = 0, \quad \frac{d}{dx}\left(-\frac{1}{2}\right) = 0, \quad \frac{d}{dx}\left(\sqrt{3}\right) = 0$ □



2.16 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

Proof of Rule 1 We apply the definition of derivative to $f(x) = c$, the function whose outputs have the constant value c (Fig. 2.16). At every value of x, we find that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0.$$ □

The next rule tells how to differentiate x^n if n is a positive integer.

> **Rule 2 Power Rule for Positive Integers**
>
> If n is a positive integer, then
>
> $$\frac{d}{dx}x^n = nx^{n-1}.$$

To apply the Power Rule, we subtract 1 from the original exponent (n) and multiply the result by n.

EXAMPLE 2

f	x	x^2	x^3	x^4	...
f'	1	$2x$	$3x^2$	$4x^3$	...

□

Proof of Rule 2 If $f(x) = x^n$, then $f(x+h) = (x+h)^n$. Since n is a positive integer, we can use the fact that

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

EXAMPLE 9 Find the derivative of $y = \dfrac{t^2-1}{t^2+1}$.

Solution We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$:

$$\begin{aligned}\frac{dy}{dt} &= \frac{(t^2+1)\cdot 2t - (t^2-1)\cdot 2t}{(t^2+1)^2} && \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}\\ &= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2}\\ &= \frac{4t}{(t^2+1)^2}.\end{aligned}$$

□

The Power Rule for Negative Integers

The Power Rule for negative integers is the same as the rule for positive integers.

> **Rule 7 Power Rule for Negative Integers**
>
> If n is a negative integer and $x \neq 0$, then
>
> $$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof of Rule 7 The proof uses the Quotient Rule in a clever way. If n is a negative integer, then $n = -m$ where m is a positive integer. Hence, $x^n = x^{-m} = 1/x^m$ and

$$\begin{aligned}\frac{d}{dx}(x^n) &= \frac{d}{dx}\left(\frac{1}{x^m}\right)\\ &= \frac{x^m\cdot\frac{d}{dx}(1) - 1\cdot\frac{d}{dx}(x^m)}{(x^m)^2} && \text{Quotient Rule with } u = 1 \text{ and } v = x^m\\ &= \frac{0 - mx^{m-1}}{x^{2m}} && \text{Since } m > 0,\ \frac{d}{dx}(x^m) = mx^{m-1}\\ &= -mx^{-m-1}\\ &= nx^{n-1}. && \text{Since } -m = n\end{aligned}$$

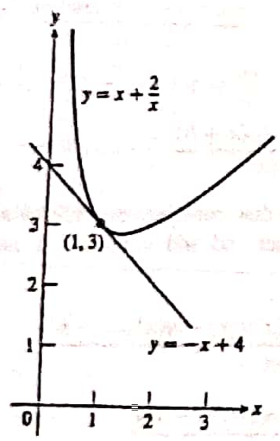
□

EXAMPLE 10

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$$

□



2.19 The tangent to the curve $y = x + (2/x)$ at (1, 3). The curve has a third-quadrant portion not shown here. We will see how to graph functions like this in Chapter 3.

EXAMPLE 11 Find an equation for the tangent to the curve

$$y = x + \frac{2}{x}$$

at the point (1, 3) (Fig. 2.19).

How to read the symbols for derivatives

y' "y prime" $\quad y''$ "y double prime"

$\dfrac{d^2y}{dx^2}$ "d squared y dx squared"

y''' "y triple prime"

$y^{(n)}$ "y super n"

$\dfrac{d^ny}{dx^n}$ "d to the n of y by dx to the n"

EXAMPLE 13 The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative:	$y' = 3x^2 - 6x$
Second derivative:	$y'' = 6x - 6$
Third derivative:	$y''' = 6$
Fourth derivative:	$y^{(4)} = 0.$

The function has derivatives of all orders, the fifth and later derivatives all being zero. ☐

Exercises 2.2

Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

1. $y = -x^2 + 3$
2. $y = x^2 + x + 8$
3. $s = 5t^3 - 3t^5$
4. $w = 3z^7 - 7z^3 + 21z^2$
5. $y = \dfrac{4x^3}{3} - x$
6. $y = \dfrac{x^3}{3} + \dfrac{x^2}{2} + \dfrac{x}{4}$
7. $w = 3z^{-2} - \dfrac{1}{z}$
8. $s = -2t^{-1} + \dfrac{4}{t^2}$
9. $y = 6x^2 - 10x - 5x^{-2}$
10. $y = 4 - 2x - x^{-3}$
11. $r = \dfrac{1}{3s^2} - \dfrac{5}{2s}$
12. $r = \dfrac{12}{\theta} - \dfrac{4}{\theta^3} + \dfrac{1}{\theta^4}$

In Exercises 13–16, find y' (a) by applying the Product Rule and (b) by multiplying the factors to produce a sum of simpler terms to differentiate.

13. $y = (3 - x^2)(x^3 - x + 1)$
14. $y = (x - 1)(x^2 + x + 1)$
15. $y = (x^2 + 1)\left(x + 5 + \dfrac{1}{x}\right)$
16. $y = \left(x + \dfrac{1}{x}\right)\left(x - \dfrac{1}{x} + 1\right)$

Find the derivatives of the functions in Exercises 17–28.

17. $y = \dfrac{2x + 5}{3x - 2}$
18. $z = \dfrac{2x + 1}{x^2 - 1}$
19. $g(x) = \dfrac{x^2 - 4}{x + 0.5}$
20. $f(t) = \dfrac{t^2 - 1}{t^2 + t - 2}$
21. $v = (1 - t)(1 + t^2)^{-1}$
22. $w = (2x - 7)^{-1}(x + 5)$
23. $f(s) = \dfrac{\sqrt{s} - 1}{\sqrt{s} + 1}$
24. $u = \dfrac{5x + 1}{2\sqrt{x}}$
25. $v = \dfrac{1 + x - 4\sqrt{x}}{x}$
26. $r = 2\left(\dfrac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$
27. $y = \dfrac{1}{(x^2 - 1)(x^2 + x + 1)}$
28. $y = \dfrac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

Find the derivatives of all orders of the functions in Exercises 29 and 30.

$\dfrac{d^4y}{dx^4}$

29. $y = \dfrac{x^4}{2} - \dfrac{3}{2}x^2 - x$
30. $y = \dfrac{x^5}{120}$

Find the first and second derivatives of the functions in Exercises 31–38. $\dfrac{x^3}{x} + \dfrac{7}{x} \Rightarrow x^2 + 7x^{-1}$

31. $y = \dfrac{x^3 + 7}{x}$
32. $s = \dfrac{t^2 + 5t - 1}{t^2}$
33. $r = \dfrac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$
34. $u = \dfrac{(x^2 + x)(x^2 - x + 1)}{x^4}$
35. $w = \left(\dfrac{1 + 3z}{3z}\right)(3 - z)$
36. $w = (z + 1)(z - 1)(z^2 + 1)$
37. $p = \left(\dfrac{q^2 + 3}{12q}\right)\left(\dfrac{q^4 - 1}{q^3}\right)$
38. $p = \dfrac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$

Using Numerical Values

39. Suppose u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

a) $\dfrac{d}{dx}(uv)$ b) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ c) $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ d) $\dfrac{d}{dx}(7v - 2u)$

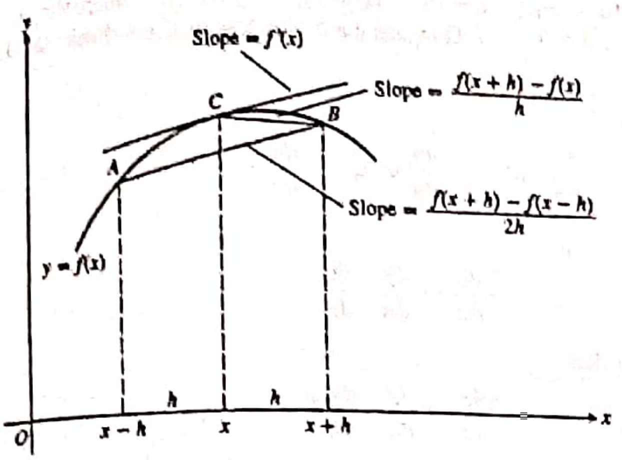
40. Suppose u and v are differentiable functions of x and that

$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

a) $\dfrac{d}{dx}(uv)$ b) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ c) $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ d) $\dfrac{d}{dx}(7v - 2u)$

See the figure below.



a) To see how rapidly the centered difference quotient for $f(x) = \sin x$ converges to $f'(x) = \cos x$, graph $y = \cos x$ together with
$$y = \frac{\sin(x+h) - \sin(x-h)}{2h}$$
over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3. Compare the results with those obtained in Exercise 69 for the same values of h.

b) To see how rapidly the centered difference quotient for $f(x) = \cos x$ converges to $f'(x) = -\sin x$, graph $y = -\sin x$ together with
$$y = \frac{\cos(x+h) - \cos(x-h)}{2h}$$
over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3. Compare the results with those obtained in Exercise 70 for the same values of h.

72. *A caution about centered difference quotients. (Continuation of Exercise 71.)* The quotient
$$\frac{f(x+h) - f(x-h)}{2h}$$
may have a limit as $h \to 0$ when f has no derivative at x. As a case in point, take $f(x) = |x|$ and calculate
$$\lim_{h \to 0} \frac{|0+h| - |0-h|}{2h}.$$
As you will see, the limit exists even though $f(x) = |x|$ has no derivative at $x = 0$.

73. Graph $y = \tan x$ and its derivative together on $(-\pi/2, \pi/2)$. Does the graph of the tangent function appear to have a smallest slope? a largest slope? Is the slope ever negative? Give reasons for your answers.

74. Graph $y = \cot x$ and its derivative together for $0 < x < \pi$. Does the graph of the cotangent function appear to have a smallest slope? a largest slope? Is the slope ever positive? Give reasons for your answers.

75. Graph $y = (\sin x)/x$, $y = (\sin 2x)/x$, and $y = (\sin 4x)/x$ together over the interval $-2 \le x \le 2$. Where does each graph appear to cross the y-axis? Do the graphs really intersect the axis? What would you expect the graphs of $y = (\sin 5x)/x$ and $y = (\sin(-3x))/x$ to do as $x \to 0$? Why? What about the graph of $y = (\sin kx)/x$ for other values of k? Give reasons for your answers.

76. *Radians vs. degrees.* What happens to the derivatives of $\sin x$ and $\cos x$ if x is measured in degrees instead of radians? To find out, take the following steps.

a) With your graphing calculator or computer grapher *in degree mode*, graph
$$f(h) = \frac{\sin h}{h}$$
and estimate $\lim_{h\to 0} f(h)$. Compare your estimate with $\pi/180$. Is there any reason to believe the limit *should* be $\pi/180$?

b) With your grapher still in degree mode, estimate
$$\lim_{h \to 0} \frac{\cos h - 1}{h}.$$

c) Now go back to the derivation of the formula for the derivative of $\sin x$ in the text and carry out the steps of the derivation using degree-mode limits. What formula do you obtain for the derivative?

d) Work through the derivation of the formula for the derivative of $\cos x$ using degree-mode limits. What formula do you obtain for the derivative?

e) The disadvantages of the degree-mode formulas become apparent as you start taking derivatives of higher order. Try it. What are the second and third degree-mode derivatives of $\sin x$ and $\cos x$?

2.5 The Chain Rule

We now know how to differentiate $\sin x$ and $x^2 - 4$, but how do we differentiate a composite like $\sin(x^2 - 4)$? The answer is, with the Chain Rule, which says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points. The Chain Rule is probably the most widely used differentiation rule in mathematics. This section describes the rule and how to use it. We begin with examples.

EXAMPLE 1 The function $y = 6x - 10 = 2(3x - 5)$ is the composite of the functions $y = 2u$ and $u = 3x - 5$. How are the derivatives of these three functions related?

Solution We have

$$\frac{dy}{dx} = 6, \qquad \frac{dy}{du} = 2, \qquad \frac{du}{dx} = 3.$$

Since $6 = 2 \cdot 3$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

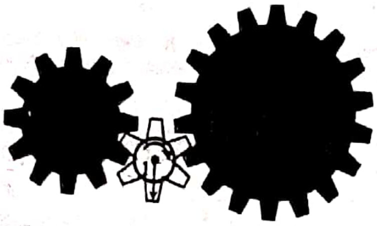
❑

Is it an accident that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}?$$

If we think of the derivative as a rate of change, our intuition allows us to see that this relationship is reasonable. For $y = f(u)$ and $u = g(x)$, if y changes twice as fast as u and u changes three times as fast as x, then we expect y to change six times as fast as x. This is much like the effect of a multiple gear train (Fig. 2.39).

Let us try this again on another function.



2.39 When gear A makes x turns, gear B makes u turns and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ and $u = 3x$, so $y = 3x/2$. Thus $dy/du = 1/2$, $du/dx = 3$, and $dy/dx = 3/2 = (dy/du)(du/dx)$.

EXAMPLE 2

$$y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$$

is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned} \frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x \\ &= 36x^3 + 12x \end{aligned}$$

and

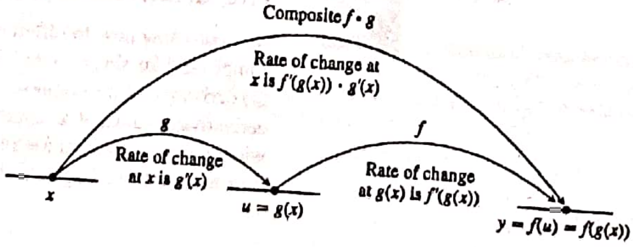
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(9x^4 + 6x^2 + 1) \\ &= 36x^3 + 12x. \end{aligned}$$

Once again,

$$\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx}.$$

❑

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x. This is known as the Chain Rule (Fig. 2.40).



2.40 Rates of change multiply: the derivative of $f \circ g$ at x is the derivative of f at the point $g(x)$ times the derivative of g at x.

Theorem 5
The Chain Rule
If $f(u)$ is differentiable at the point $u = g(x)$, and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x). \tag{1}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \tag{2}$$

where dy/du is evaluated at $u = g(x)$.

It would be tempting to try to prove the Chain Rule by writing

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

and taking the limit as $\Delta x \to 0$. This would work if we knew that Δu, the change in u, was nonzero, but we do not know this. A small change in x could conceivably produce no change in u. The proof requires a different approach, using ideas in Section 3.7. We will return to it when the time comes.

EXAMPLE 3 Find the derivative of $y = \sqrt{x^2 + 1}$.

Solution Here $y = f(g(x))$, where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$. Since the derivatives of f and g are

$$f'(u) = \frac{1}{2\sqrt{u}} \qquad \text{and} \qquad g'(x) = 2x,$$

the Chain Rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x) \\ &= \frac{1}{2\sqrt{g(x)}} \cdot g'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot (2x) \\ &= \frac{x}{\sqrt{x^2+1}}.\end{aligned}$$

❑

The "Outside-Inside" Rule

It sometimes helps to think about the Chain Rule the following way. If $y = f(g(x))$, Eq. (2) tells us that

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x). \tag{3}$$

In words, Eq. (3) says: To find dy/dx, differentiate the "outside" function f and leave the "inside" $g(x)$ alone; then multiply by the derivative of the inside.

$$3s^2\frac{ds}{dt} = -6ks^2$$

$$\frac{ds}{dt} = -2k.$$

The side length is *decreasing* at the constant rate of $2k$ units per hour. Thus, if the initial length of the cube's side is s_0, the length of its side one hour later is $s_1 = s_0 - 2k$. This equation tells us that

$$2k = s_0 - s_1.$$

The melting time is the value of t that makes $2kt = s_0$. Hence,

$$t_{\text{melt}} = \frac{s_0}{2k} = \frac{s_0}{s_0 - s_1} = \frac{1}{1 - (s_1/s_0)}.$$

But

$$\frac{s_1}{s_0} = \frac{\left(\frac{3}{4}V_0\right)^{1/3}}{(V_0)^{1/3}} = \left(\frac{3}{4}\right)^{1/3} \approx 0.91.$$

Therefore,

$$t_{\text{melt}} = \frac{1}{1 - 0.91} \approx 11 \text{ h}.$$

If 1/4 of the cube melts in 1 h, it will take about 10 h more for the rest of it to melt. □

If we were natural scientists interested in testing the assumptions on which our mathematical model is based, our next step would be to run a number of experiments and compare their outcomes with the model's predictions. One practical application might lie in analyzing the proposal to tow large icebergs from polar waters to offshore locations near southern California, where the melting ice could provide fresh water. As a first approximation, we might imagine the iceberg to be a large cube or rectangular solid, or perhaps a pyramid. We will say more about mathematical modeling in Section 4.2.

Exercises 2.5

Derivative Calculations

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9, \quad u = (1/2)x^4$
2. $y = 2u^3, \quad u = 8x - 1$
3. $y = \sin u, \quad u = 3x + 1$
4. $y = \cos u, \quad u = -x/3$
5. $y = \cos u, \quad u = \sin x$
6. $y = \sin u, \quad u = x - \cos x$
7. $y = \tan u, \quad u = 10x - 5$
8. $y = -\sec u, \quad u = x^2 + 7x$

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In Exercises 9–18, write the function in the form $y=f(u)$ and $u=g(x)$. Then find dy/dx as a function of x.

9. $y=(2x+1)^5$

10. $y=(4-3x)^9$

11. $y=\left(1-\frac{x}{7}\right)^{-7}$

12. $y=\left(\frac{x}{2}-1\right)^{-10}$

13. $y=\left(\frac{x^2}{8}+x-\frac{1}{x}\right)^4$

14. $y=\left(\frac{x}{5}+\frac{1}{5x}\right)^5$

15. $y=\sec(\tan x)$

16. $y=\cot\left(\pi-\frac{1}{x}\right)$

17. $y=\sin^3 x$

18. $y=5\cos^{-4} x$

Find the derivatives of the functions in Exercises 19–38.

19. $p=\sqrt{3-t}$

20. $q=\sqrt{2r-r^2}$

21. $s=\frac{4}{3\pi}\sin 3t+\frac{4}{5\pi}\cos 5t$

22. $s=\sin\left(\frac{3\pi t}{2}\right)+\cos\left(\frac{3\pi t}{2}\right)$

23. $r=(\csc\theta+\cot\theta)^{-1}$

24. $r=-(\sec\theta+\tan\theta)^{-1}$

25. $y=x^2\sin^4 x+x\cos^{-2} x$

26. $y=\frac{1}{x}\sin^{-5} x-\frac{x}{3}\cos^3 x$

27. $y=\frac{1}{21}(3x-2)^7+\left(4-\frac{1}{2x^2}\right)^{-1}$

28. $y=(5-2x)^{-3}+\frac{1}{8}\left(\frac{2}{x}+1\right)^4$

29. $y=(4x+3)^4(x+1)^{-3}$

30. $y=(2x-5)^{-1}(x^2-5x)^6$

31. $h(x)=x\tan(2\sqrt{x})+7$

32. $k(x)=x^2\sec\left(\frac{1}{x}\right)$

33. $f(\theta)=\left(\frac{\sin\theta}{1+\cos\theta}\right)^2$

34. $g(t)=\left(\frac{1+\cos t}{\sin t}\right)^{-1}$

35. $r=\sin(\theta^2)\cos(2\theta)$

36. $r=\sec\sqrt{\theta}\tan\left(\frac{1}{\theta}\right)$

37. $q=\sin\left(\frac{t}{\sqrt{t+1}}\right)$

38. $q=\cot\left(\frac{\sin t}{t}\right)$

In Exercises 39–48, find dy/dt.

39. $y=\sin^2(\pi t-2)$

40. $y=\sec^2\pi t$

41. $y=(1+\cos 2t)^{-4}$

42. $y=(1+\cot(t/2))^{-2}$

43. $y=\sin(\cos(2t-5))$

44. $y=\cos\left(5\sin\left(\frac{t}{3}\right)\right)$

45. $y=\left(1+\tan^4\left(\frac{t}{12}\right)\right)^3$

46. $y=\frac{1}{6}\left(1+\cos^2(7t)\right)^3$

47. $y=\sqrt{1+\cos(t^2)}$

48. $y=4\sin\left(\sqrt{1+\sqrt{t}}\right)$

Find y'' in Exercises 49–52.

49. $y=\left(1+\frac{1}{x}\right)^3$

50. $y=(1-\sqrt{x})^{-1}$

51. $y=\frac{1}{9}\cot(3x-1)$

52. $y=9\tan\left(\frac{x}{3}\right)$

Finding Numerical Values of Derivatives

In Exercises 53–58, find the value of $(f\circ g)'$ at the given value of x.

53. $f(u)=u^5+1,\quad u=g(x)=\sqrt{x},\quad x=1$

54. $f(u)=1-\frac{1}{u},\quad u=g(x)=\frac{1}{1-x},\quad x=-1$

55. $f(u)=\cot\frac{\pi u}{10},\quad u=g(x)=5\sqrt{x},\quad x=1$

56. $f(u)=u+\frac{1}{\cos^2 u},\quad u=g(x)=\pi x,\quad x=1/4$

57. $f(u)=\frac{2u}{u^2+1},\quad u=g(x)=10x^2+x+1,\quad x=0$

58. $f(u)=\left(\frac{u-1}{u+1}\right)^2,\quad u=g(x)=\frac{1}{x^2}-1,\quad x=-1$

59. Suppose that functions f and g and their derivatives with respect to x have the following values at $x=2$ and $x=3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	−3
3	3	−4	2π	5

Find the derivatives with respect to x of the following combinations at the given value of x.

a) $2f(x),\quad x=2$
b) $f(x)+g(x),\quad x=3$
c) $f(x)\cdot g(x),\quad x=3$
d) $f(x)/g(x),\quad x=2$
e) $f(g(x)),\quad x=2$
f) $\sqrt{f(x)},\quad x=2$
g) $1/g^2(x),\quad x=3$
h) $\sqrt{f^2(x)+g^2(x)},\quad x=2$

60. Suppose that the functions f and g and their derivatives with respect to x have the following values at $x=0$ and $x=1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	−4	−1/3	−8/3

Find the derivatives with respect to x of the following combinations at the given value of x.

a) $5f(x)-g(x),\quad x=1$
b) $f(x)g^3(x),\quad x=0$
c) $\frac{f(x)}{g(x)+1},\quad x=1$
d) $f(g(x)),\quad x=0$
e) $g(f(x)),\quad x=0$
f) $(x^{11}+f(x))^{-2},\quad x=1$
g) $f(x+g(x)),\quad x=0$